Generalized Likelihood Test for FDI in Redundant Sensor Configurations

Kevin C. Daly,* Eliezer Gai,† and James V. Harrison‡
The Charles Stark Draper Laboratory, Inc., Cambridge, Mass.

The Generalized Likelihood Test (GLT) approach to failure detection and isolation (FDI) in redundant sets of inertial sensors is described and its relationship to the overall FDI structure is explored. Specific formulations of both the detection and the isolation problems are presented. The performance of the resulting FDI system is analyzed by means of the second-order statistics of the detection and isolation decision functions. The equivalence of the GLT approach to several previously reported approaches for both single-degree-of-freedom (SDOF) and two-degree-of-freedom (TDOF) sensors is described. To illustrate its application, the GLT approach is used to compare the FDI performance of three different redundant sensor configurations; a conical configuration of five SDOF sensors, a dodecahedron configuration of six SDOF sensors, and an octahedron configuration of four TDOF sensors.

Introduction

IN many critical aerospace applications, the reliability requirements placed on the inertial measurement function exceed those which can be met with a single string design using available sensors. Inertial measurement systems incorporating redundant sensors may realize significant reliability improvement relative to single string systems, provided that the redundant sensor configurations achieve a high degree of fault tolerance. This fault tolerance may be achieved passively through the application of data selection techniques[1] or actively through the use of failure detection and isolation (FDI) techniques.[2]

This paper presents an FDI approach which is based on the concept of the generalized likelihood test (GLT). Several applications of the GLT to inertial sensor failure detection and isolation have been reported in the literature. Willsky and Jones[3] developed a sequential approach based on the generalized likelihood ratio test. Carlson[4] proposed a single sample approach for a dual inertial platform application. Wilcox[5] defined an approach which uses the likelihood function but in a manner different from the GLT formulation.

In the FDI approach described herein, the failure detection and isolation problems are formulated as composite hypothesis testing problems.[6] The concept of the generalized likelihood test GLT is used to derive detection and isolation decision functions. The implementation of these decision functions is discussed for both single degree-of-freedom (SDOF) and two degree-of-freedom (TDOF) sensor configurations.

The GLT approach to FDI is shown to provide performance which is equivalent to other commonly used methods[7,8] while requiring less computer resources for implementation. Applications to three representative sensor geometries – a dodecahedron array of six SDOF gyroscopes, an octahedron array of four TDOF gyroscopes, and a pentad array of five SDOF sensors – are described, and the detection and isolation performance of the decision functions is evaluated using a Monte Carlo simulation.

Received Feb. 27, 1978; revision received June 15, 1978.

Index categories: Guidance and Control; Sensor Systems; Spacecraft Navigation, Guidance, and Flight-Path Control.

*Staff Member, Control and Flight Dynamics Division. Member AIAA.

†Staff Member, Inertial Division.

‡Staff Member, System Engineering Division.

Failure Detection and Isolation

FDI Structure

The FDI structure may be decomposed into three major components: data processing, decision functions, and threshold testing. A set of redundant measurements is processed to obtain a set of residuals which measure the inconsistency among individual sensor outputs. These residuals are the inputs to decision functions which can be tested against appropriate thresholds to make FDI decisions. The structure is quite general. The following paragraphs briefly discuss the broad spectrum of approaches which are encompassed by this structure.

Consider the data processing function. In a minimum processing approach, uncompensated sensor outputs are processed immediately through a set of parity equations to generate a set of residuals which are independent of the true measurement but reflect the measurement noise and any failure effects which might be present. Additional processing may be included to provide for measurement compensation or modification of the sensor data signal-to-noise characteristics. Single sample or finite memory data processing techniques may be used. The extent of the processing done prior to making FDI decisions is limited by the allowable time to detect and isolate sensor failures as well as by the availability of computer resources. The allowable time to detect and isolate sensor failures depends on the magnitudes of the failures and the sensitivity of system performance to these magnitudes.

The decision function also permits a wide spectrum of choices. Although a single decision function may be used to simultaneously detect and isolate sensor failures,[9] the detection decision and the isolation decision generally are made separately. This separation proves convenient because the relatively simple detection decision must be made each time sensor data are read, whereas the more complex isolation decision is made only when detection occurs. The decision functions may be functions of single samples or multiple samples of the measurement data. The reliable detection and isolation of small failure magnitudes relative to the ambient sensor and measurement noise generally require decision functions which incorporate more than one sample.[10,11] Single sample tests, on the other hand, often prove adequate for detecting and isolating larger failure magnitudes.[12]

The FDI decision process requires the determination of boundaries between the regions of normal sensor performance and failed sensor performance. These boundaries are usually manifested in thresholds against which the decision functions are tested to make FDI decisions. Thresholds are commonly determined by relating the effects of failure magnitudes during the course of the mission to mission performance.[13] Alternatively, thresholds can be determined by defining boundaries on normal sensor behavior. A representative decision tree associated with the FDI decision structure is indicated in Fig. 1. In this diagram the conditional probability of taking each path from a branch point is shown on the vertical path from the branch point while the joint probability of being on a given branch is shown on the horizontal path. To preserve the common use, the conditional probability of detection (given that a failure has occurred) is called the probability of detection (P_D) and the conditional probability of false alarm (given that no failure has occurred) is called the probability of false alarm (P_{FA}).

The seven terminal states of Fig. 1 completely specify the outcomes of the FDI decision structure. FDI performance may be assessed by the probabilities of the various successful (correct isolation, correct rejection, and correct operation) and unsuccessful (wrong isolation, missed isolation, missed detection, and false isolation) outcomes. The overall risk associated with FDI decisions may be denoted $R(D_D, D_I)$ where D_D is the detection decision structure and D_I is the isolation decision structure. This risk is given by

$$R(D_D,D_I) = \sum_{i=1}^{7} C_i P_i$$

where C_i is the cost associated with outcome i, and P_i is the probability associated with outcome i. For a given set of underlying sensor failure statistics, the risk may be adjusted by changing the detection and isolation decision structure.

The preceding paragraphs suggest the broad spectrum of choices available to the FDI system designer. The discussion in this paper is limited to single sample tests made at the sensor level as opposed to the state level. As such, the approach here is primarily intended for applications in which the failure magnitudes to be detected and isolated are relatively large. Separate detection and isolation decision stages are assumed. The emphasis is on the choice of appropriate decision functions. The methodology for choosing appropriate thresholds is not addressed.

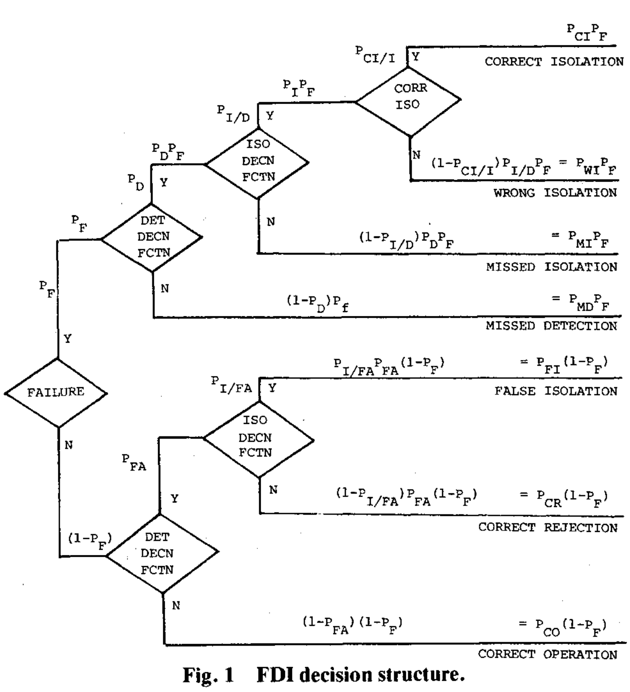

Fig. 1 FDI decision structure.

Parity Equations

Assume that a set of redundant inertial sensors yields n measurements. These measurements can be represented as the n dimensional vector $\boldsymbol{m}$. In the absence of sensor failures, the three-dimensional true inertial quantity $\boldsymbol{x}$, which the sensors measure, is related to $\boldsymbol{m}$ through the measurement equation:

$$\boldsymbol{m} = H\boldsymbol{x} + \epsilon \tag{1}$$

The rows of the geometry matrix H consist of the direction cosines of the n-measurement axes with respect to an arbitrarily chosen reference frame. ϵ is an n-dimensional measurement noise vector which is assumed to be Gaussian with zero mean and covariance matrix R. In order to detect sensor failures, a function of the measurement vector $\boldsymbol{m}$ is defined which is independent of the true inertial quantity $\boldsymbol{x}$. Let $\boldsymbol{v}_i$ be an n-dimensional vector of constants such that

$$\boldsymbol{v}_i^T H = 0 \tag{2}$$

Then the scalar function $p_i(\boldsymbol{m})$ given by

$$p_i(\boldsymbol{m}) = \boldsymbol{v}_i^T \boldsymbol{m} = \boldsymbol{v}_i^T \epsilon \tag{3}$$

is, in the absence of sensor failures, strictly a function of the measurement noise. Equation (3) is referred to as a parity equation. Since ϵ was assumed to be Gaussian with zero mean and covariance matrix R, it follows that the parity equation residual p_i is a Gaussian random variable with zero mean and variance

$$\sigma_{pi}^2 = \boldsymbol{v}_i^T R \boldsymbol{v}_i \tag{4}$$

Sensor failures are modeled as a bias shift of magnitude b and arbitrary sign. Although somewhat restrictive, this model has been found to be adequate for a broad class of sensor failures.[14] In the presence of a failure of the jth sensor, the measurement equation takes the form:

$$\boldsymbol{m} = H\boldsymbol{x} + \epsilon + \boldsymbol{a}_j b \tag{5}$$

where all components of $\boldsymbol{a}_j$ are zero except the jth component which is 1. The scalar b represents the sign and magnitude of the failure. The parity equation then takes the form:

$$p_i(\boldsymbol{m}) = \boldsymbol{v}_i^T \epsilon + \boldsymbol{v}_i^T \boldsymbol{a}_j b = \boldsymbol{v}_i^T \epsilon + v_{ij} b \tag{6}$$

where v_{ij} is the jth component of $\boldsymbol{v}_i$. In the presence of a sensor failure, therefore, the parity equation residual is a Gaussian random variable with nonzero mean $v_{ij}b$ and variance given by Eq. (4). It is the difference in the means of the parity equation residual in the absence and presence of sensor failures which provides a basis for failure detection.

If a vector $\boldsymbol{v}_i$ satisfying Eq. (2) exists such that all of its elements are nonzero, then failure detection can be accomplished with a single parity equation. The isolation of a failure to a failed sensor, however, requires a set of parity equations. This set of parity equations can be represented in terms of a matrix V, the rows of which are the parity equation coefficients $\boldsymbol{v}_i^T$ of the individual parity equations. Under the constraint of Eq. (2), it follows that:

$$VH = 0 \tag{7}$$

and the vector $\boldsymbol{p}$ of parity equation residuals is given by

$$\boldsymbol{p} = V\boldsymbol{m} \tag{8}$$

Although the row dimension of the matrix V is arbitrary, there are at most $n-3$ linearly independent parity equations.[1]

Generalized Likelihood Approach to Failure Detection and Isolation

Generalized Likelihood Test Formulation

Consider a set of n SDOF sensors. The problems of detecting and isolating sensor failures fall within the general framework of composite hypothesis tests.[6] The detection decision, for example, can be viewed as a choice between two hypotheses concerning the presence or absence of a failure. Since the failure magnitude is completely unknown, the detection problem is regarded as a composite binary hypothesis test. Once a failure has been detected, the isolation decision involves a choice among n hypotheses, one associated with each of the n sensors. The isolation problem is therefore regarded as a composite multiple hypothesis test.[15]

If the failure magnitude and sign were known a priori, the detection and isolation problems could be formulated as simple hypothesis tests for which the likelihood ratio test is known to be most powerful.[6] In the case of hypothesis tests with unknown parameters which can assume both negative and positive values, however, a uniformly most powerful test does not exist. An alternative test, which provides satisfactory results in most cases, is the GLT.[6] The GLT is mechanized by substituting the maximum likelihood estimate of the unknown parameters for each hypothesis into the likelihood ratio test, and then treating the composite hypothesis test as a simple hypothesis test.

Formulation of the Detection Problem

Given the vector p of parity equation residuals, the two hypotheses to be tested are identified as H_0, the normal mode, and H_1 the failure mode. In H_1 the bias failure model is assumed, with the bias failure magnitude and sign being completely unknown. For the case of uncorrelated measurement noise with unit variance, the statistics of p under the alternative hypotheses are:

$$H_0{:}E[p]=0 \qquad E[pp^T]=VV^T \tag{9}$$

$$H_1{:}E[p]=\mu \qquad E[(p-\mu)(p-\mu)^T]=VV^T \tag{10}$$

where μ, the mean in the failure mode, can take on both negative and positive values.

Since p is a Gaussian random vector, the log likelihood ratio $\Lambda(p)$ for the two hypotheses is given by:

$$\Lambda(p)=\tfrac{1}{2}[p^T(VV^T)^{-1}p-(p-\mu)^T(VV^T)^{-1}(p-\mu)] \tag{11}$$

The maximum likelihood estimate $\hat{\mu}$ of μ is the value which maximizes the expression

$$[-\tfrac{1}{2}(p-\mu)^T(VV^T)^{-1}(p-\mu)]$$

Clearly,

$$\hat{\mu}=p \tag{12}$$

Substituting this result into the expression for $\Lambda(p)$ yields the detection decision function

$$DF_D=p^T(VV^T)^{-1}p \tag{13}$$

Formulating the detection problem as a composite binary hypothesis test thus leads to a weighted squared-error decision function for detection.

Formulation of the Isolation Problem

Failure isolation is initiated after the detection decision function has crossed the decision threshold. Given the vector p of parity equation residuals, the n hypotheses to be tested are identified as:

$$H_j{:}\ \text{a bias failure has occured in the } j\text{th sensor} \tag{14}$$

where $j=1,2,...,n$. For the case of uncorrelated measurement noise with unit variance, the statistics of p under the alternative hypotheses are:

$$H_j{:}E[p]=v_jb \qquad E[pp^T]=VV^T \tag{15}$$

where v_j is the jth column of the matrix V. Since p is a Gaussian random vector, the associated likelihood function is:

$$\Delta_j=K\exp\left\{-\frac{1}{2}(p-v_jb)^T(VV^T)^{-1}(p-v_jb)\right\} \tag{16}$$

where K is a constant. The failure magnitude b can take on both negative and positive values. Under hypothesis H_j, the maximum likelihood estimate $\hat{b}$ of b is

$$\hat{b}=\frac{p^T(VV^T)^{-1}v_j}{v_j^T(VV^T)^{-1}v_j} \tag{17}$$

Substituting $\hat{b}$ into the log of the likelihood function, and simplifying the result, yields the isolation decision function under the jth hypothesis

$$DF_{I_j}=\frac{(p^T(VV^T)^{-1}v_j)^2}{v_j^T(VV^T)^{-1}v_j} \tag{18}$$

Given p, a failed sensor is identified by computing the n values of the decision function. If DF_{I_k} is the largest of these n values, then the kth sensor is the one which is most likely to have failed.

Choice of a V Matrix

The matrix V completely defines the vector p of parity equation residuals given by Eq. (8). Subject only to the constraint of Eq. (7) imposed by the geometrical arrangement of the sensors, the FDI designer is free to choose a V matrix to satisfy his design requirements. To that end, two constraints are imposed on the particular choice of a V matrix to implement the detection and isolation decision functions presented as Eqs. (13) and (18).

The first constraint on the choice of V is that it shall have $n-3$ rows which are linearly independent. The linear independence of the rows of V guarantees the invertability of VV^T as required in both decision functions. $n-3$ rows are specified because the use of fewer than $n-3$ linearly independent parity equations does not take advantage of all of the information which is potentially available for making FDI decisions. The use of more than $n-3$ parity equations, on the other hand, yields no additional information for making such decisions. When more than $n-3$ parity equations are used, the statistics of p, upon which the FDI decisions depend, can always be expressed in terms of the statistics of another vector p' derived from a subset of linearly independent equations.

The second constraint on the choice of V is that

$$VV^T=I \tag{19}$$

This constraint results in a simplification of the decision functions given in Eqs. (13) and (18). In simplified form, the decision function for detection is simply the total squared-error decision function

$$DF_D=p^Tp \tag{20}$$

and the decision function for isolation under the jth hypothesis is:

$$DF_{I_j}=\frac{(p^Tv_j)^2}{v_j^Tv_j} \tag{21}$$

A further simplification of the isolation decision function is realized for particular sensor geometries for which the dot product $v_j^T v_j$ does not depend on the index j. This property is referred to as uniform detectability.[7] The dodecahedron, the octahedron, and the pentad sensor configurations described in the subsequent section are examples of such sensor geometries. In those cases, the isolation decision function under the jth hypothesis is reduced to:

$$DF_{I_j} = (p^T v_j)^2 \tag{22}$$

Failure isolation using this decision function permits a simple geometrical interpretation. In the $n-3$ dimensional parity space the jth sensor can be uniquely associated with the vector direction v_j. If we then identify sensor k as the failed sensor whose associated direction v_k makes the smallest angle with the vector p of parity equation residuals, we have a decision rule which is entirely equivalent to the one stated following Eq. (18).

A consequence of Eq. (7) and the two constraints imposed on the choice of V is that

$$V^T V = I - H(H^T H)^{-1} H^T \tag{23}$$

The proof of this assertion was provided by Potter.[16] Let the square matrices A and B be defined as:

$$A = \begin{bmatrix} (H^T H)^{-1} H^T \\ \hline V \end{bmatrix} \tag{24}$$

$$B = (H \;\vdots\; V^T) \tag{25}$$

Then clearly $AB = I$. Since the left and right inverses of square finite matrices are equal, it follows that $BA = I$, which establishes Eq. (23).

The right-hand side of Eq. (23) is an $n \times n$ matrix which satisfies Eq. (7)

$$[I - H(H^T H)^{-1} H^T] H = 0 \tag{26}$$

and, therefore, can be interpreted as a matrix V_n of coefficients for n parity equations. The parity equation, Eq. (8), becomes

$$p' = V_n m = [I - H(H^T H)^{-1} H^T] m = m - \hat{m} \tag{27}$$

where $\hat{m}$ is the least-squares estimate (LSE) of m. Thus p' is an n-dimensional vector of parity equation residuals obtained by subtracting the LSE of each measurement based on all available measurements from the actual measurement.

A convenient algorithm for computing a V matrix which satisfies the constraints previously defined is given by Potter.[1]

Detection and Isolation Performance

Consider the detection decision function

$$DF_D = p^T p \tag{28}$$

In the absence of a sensor failure, DF_D has a density function which is central χ^2 with $n-3$ degrees of freedom. In the presence of a failure of the jth sensor, the density function is noncentral χ^2 with $n-3$ degrees of freedom and a noncentrality parameter

$$\gamma_j = v_j^T v_j b^2 \tag{29}$$

The parameter γ_j is an indicator of the achievable performance for the detection decision. For a fixed threshold, the probability of error for the detection decision decreases monotonically as γ_j increases. Given a failure of the jth sensor, therefore, detection performance depends not only on the failure magnitude b but also on the scalar $v_j^T v_j$, which is the jth diagonal element of the matrix $V^T V$. Since this scalar magnitude can be different for different j, the figure of merit for detection for a fixed failure magnitude is defined as the min. $v_j^T v_j$ which represents the worst case. It has been shown[7] that

$$\sum_{j=1}^{n} v_j^T v_j = n-3 \tag{30}$$

so that the best detection performance is achieved when

$$v_j^T v_j = \frac{n-3}{n} \quad \text{for all} \quad j \tag{31}$$

i.e., when the sensor geometry yields uniform detectability for all sensor failures.

Consider the isolation decision function

$$DF_{I_j} = (p^T v_j)^2 \tag{32}$$

The analytical form for the density function for DF_{I_j} consists of an infinite series of χ^2 densities of increasing degrees of freedom. The mean value of DF_{I_j} in the failure mode can be used as an indicator of the achievable performance for the isolation decision. If a bias failure of the kth sensor is assumed, the expectation of DF_{I_j} can be evaluated for uncorrelated measurement noise with unit variance:

$$E[DF_{I_j} | H_k] = v_j^T v_j + b^2 (v_k^T v_j)^2 \tag{33}$$

Since the GLT isolation algorithm will select the sensor with the largest DF_{I_j}, good isolation performance requires that the right-hand side of Eq. (33) be large if $k=j$ and small if $k \neq j$. The quantities $v_k^T v_j$ and $v_j^T v_j$ are the elements of the matrix $V^T V$, so for best performance each diagonal element should be as large as possible with respect to the other elements on its respective row. One way to design a matrix $V^T V$ with this property is to let

$$V^T V = \begin{bmatrix} a_1^T \\ a_2^T \\ \vdots \\ a_n^T \end{bmatrix} \tag{34}$$

and choose the elements of each row i so that the diagonal element a_{ii} is unity and the off-diagonal elements minimize the cost

$$C_i = \sum_{\substack{j=1 \\ j \neq i}}^{n} a_{ij}^2 \tag{35}$$

under the constraint that

$$a_i^T H = 0 \tag{36}$$

It has been shown[7] that the matrix $V^T V$ which satisfies Eqs. (34-36) is:

$$V^T V = I - H(H^T H)^{-1} H^T \tag{37}$$

Since Eq. (37) is a consequence of the two constraints imposed on the choice of a V matrix, these constraints insure good isolation performance.

Specialization to TDOF Sensors

The previous discussion assumes a set of n SDOF sensors. The extension to TDOF sensors requires certain modification to reflect the characteristics of such sensors. The assumption made in the case of SDOF sensors that the measurement noise is uncorrelated must be reconsidered. Correlation between the noise present in the two measurements derived from a TDOF sensor is possible. Assuming unit variance and a correlation coefficient, the covariance matrix R then has the form

$$R=\begin{bmatrix} \begin{matrix}1 & \rho\\ \rho & 1\end{matrix} & 0 & & \\ 0 & \begin{matrix}1 & \rho\\ \rho & 1\end{matrix} & & \\ & & \ddots & \\ & & & \begin{matrix}1 & \rho\\ \rho & 1\end{matrix} \end{bmatrix} \tag{38}$$

A practical difficulty lies in determining the values of ρ for a particular sensor. The FDI designer may respond to this difficulty in two ways. One approach is to assume $\rho=0$, design the FDI algorithms accordingly, and examine through simulations the degradation of FDI performance which occurs in the presence of nonzero values of ρ. This approach leads to the simplest algorithms and would be preferred if the performance penalty incurred for nonzero values of ρ were acceptably small. The alternative approach is to measure ρ through instrument tests and design the FDI algorithms for known correlation. The algorithms are then more complex, but should yield improved performance if the measured ρ and the true ρ are nearly the same. The authors' experience is that the former approach is perferred. The data presented in the application section of this paper support this conclusion.

Only the GLT algorithms for the case of $\rho=0$, therefore, are presented here. In that case, the detection problem formulation is not changed, and the appropriate decision function is given by Eq. (20).

In formulating the isolation problem, another characteristic of TDOF sensors must be considered. A TDOF sensor failure may be reflected in either or both of its measurement axes. In practice, a failure observed in either axis is sufficient to disqualify the data from both of the sensor axes. Thus, isolation to a failed sensor rather than to a failed axis is sufficient. The isolation problem then involves testing only $n/2$ hypotheses. The GLT decision function for isolation which corresponds to Eq. (21) is:

$$DF_I=p^T V_j(V_j^T V_j)^{-1} V_j^T p \qquad j=1,2,\ldots,n/2 \tag{39}$$

where $V_j=[v_{2j-1}, v_{2j}]$ and v_{2j-1}, v_{2j} are the two columns of the V matrix associated with TDOF sensor j. For those sensor geometries for which $V_j^T V_j$ is the same matrix for all j, which is analogous to requiring uniform detectability for the SDOF sensor case, the following simplification of the decision function for isolation is achieved

$$DF_{I_j}=(p^T v_{2j-1})^2+(p^T v_{2j})^2 \qquad j=1,2,\ldots,n/2 \tag{40}$$

This decision function is the analog of that presented in Eq. (22) for SDOF sensors.

Equivalence of GLT Approach to Other Approaches

As previously stated, the GLT algorithms are functions of a vector p_{n-3} of parity equation residuals derived from $n-3$ linearly independent parity equations. Other approaches to the design of FDI algorithms based on n parity equations have been taken. Several of these approaches are described here, and their equivalence to the GLT approach in terms of FDI performance is demonstrated. Concluding remarks concerning the author's preference for the GLT approach are appended.

The total squared error (TSE)/maximum squared residual (MSR) approach to FDI associates one parity equation residual with each of the n sensors

$$p_i=m_i-\hat{m}_i \qquad i=1,2,\ldots,n \tag{41}$$

where $\hat{m}_i$ is the least-squares estimate of m_i based upon all available measurements. Thus the n dimensional parity vector p_n is given by

$$p_n=[I-H(H^TH)^{-1}H^T]m=V_n m \tag{42}$$

Detection is based on the TSE decision function

$$DF_D=p_n^T p_n \tag{43}$$

Isolation is based on the MSR decision function

$$DF_{I_n j}=p_{n_j}^2 \tag{44}$$

where p_{n_j} is the jth component of p_n. Given detection, the failure is isolated to the sensor having the largest squared residual. For the GLT approach,

$$p_{n-3}=V_{n-3}m \tag{45}$$

where V_{n-3} satisfies the constraints imposed on the choice of a V matrix. Multiplying both sides of Eq. (45) by V_{n-3}^T, and using the results of Eqs. (23) and (42) yields

$$V_{n-3}^T p_{n-3}=[I-H(H^TH)^{-1}H^T]m=p_n \tag{46}$$

From Eqs. (43) and (13), it follows that

$$p_n^T p_n=p_{n-3}^T p_{n-3} \tag{47}$$

Thus, the decision function for detection given by Eq. (43) for the TSE/MSR approach to FDI is equivalent to the decision function for detection under the GLT approach. Furthermore, Eq. (45) implies that

$$p_{n_j}=v_j^T p_{n-3} \tag{48}$$

where v_j is the jth column of V_{n-3}. Thus, the decision function for isolation under the TSE/MSR approach is equivalent to the decision function for isolation under the GLT approach for those sensor configurations providing uniform detectability.

An FDI approach that is equivalent to the TSE/MSR approach, but with a differently derived $n\times n$ V matrix, was discussed in Ref. 7. To derive the n parity equations, assume that the ith parity equation is given by:

$$\tilde{p}_i=(v_{i1}, v_{i2}, v_{i3},\ldots v_{in})m \tag{49}$$

The parity equation residual $\tilde{p}_i$ is used to isolate failures of the ith sensor using the MSR decision function. In order to maximize the sensitivity of this equation to failures of the ith sensor, v_{ii} is constrained to be unity and the remaining parity equation coefficients are chosen to minimize the cost

$$C_i=\sum_{\substack{j=1\\ j\neq i}}^{n} v_{ij}^2 \tag{50}$$

The solution to this minimization problem is the following parity equation[7]:

$$\tilde{p}_i=m_i-\tilde{m}_i \tag{51}$$

where $\tilde{m}_i$ is the least-squares estimate of m_i based upon all available measurements except m_i. Thus,

$$\tilde{m}_i = h_i^T (H_{n-1}^T H_{n-1})^{-1} H_{n-1}^T m_{n-1} \tag{52}$$

where h_i^T is the ith row of the geometry matrix H, H_{n-1} is the H matrix excluding the ith row, and m_{n-1} is the measurement vector excluding the ith measurement. The relationship between $\hat{m}_i$ in Eq. (41) and $\tilde{m}$ in Eq. (52) is:

$$\hat{m}_i = (1+\Gamma_i^{-2})^{-1}(m_i + \Gamma_i^{-2}\tilde{m}_i) \tag{53}$$

where

$$\Gamma_i^2 = h_i^T (H_{n-1}^T H_{n-1})^{-1} h_i \tag{54}$$

Substituting this result into Eq. (41) yields

$$\tilde{p}_i = (\Gamma_i^2 + 1) p_i \tag{55}$$

The ith parity equation for the maximum detectability approach, therefore, differs from the ith parity equation for the TSE/MSR approach only by the multiplicative constant. Multiplication of a parity equation by a constant does not change the FDI performance. The maximum detectability approach, therefore, is equivalent to the TSE/MSR approach and thus is also equivalent to the GLT approach.

Specializing the TSE/MSR and maximum detectability approaches to a set of $n/2$ TDOF sensors requires certain modifications in the method for constructing the parity equation sets in the associated decision functions. Since TDOF sensor failures may be observed in both of the sensor's measurement axes, the parity equation associated with one measurement axis of a TDOF sensor is designed to be independent of the other measurement derived from that sensor. In the TSE/MSR approach, this is achieved by defining a set of parity equations:

$$p_i = m_i - \overset{*}{m}_i \tag{56}$$

where $\overset{*}{m}_i$ is taken to be the least-squares estimate of m_i based on the $n-2$ measurements that do not include the two measurements of sensor i. In the maximum detectability approach, this is achieved by constraining v_{ii} of Eq. (49) to be unity, constraining the coefficient associated with the other measurement axis of sensor i to be zero, and choosing the remaining coefficients to minimize the sum of their squared magnitudes. The decision function for detection is unchanged. The MSR decision function for isolation is modified to reflect the dual failure characteristic of TDOF sensors.

$$DF_{I_j} = p_{2j-1}^2 + p_{2j}^2 \qquad \text{where} \qquad j = 1,2,\ldots,n/2 \tag{57}$$

By obtaining the relationship between $\overset{*}{m}_i$ in Eq. (56) and m_i in Eq. (41), it can then be shown that for sensor geometry matrices which satisfy the constraint:

$$H^T H = \alpha I \tag{58}$$

where α is a constant, the TSE/MSR apprach, the maximum detectability approach, and the GLT approach are equivalent in their TDOF sensor formulations.

Since the FDI performance achievable with the GLT approach is equivalent to the FDI performance, which can be obtained with the other approaches discussed here, the choice of a particular design approach can be based on other considerations. Among these, the primary consideration is the relative complexity of the various approaches and the computer resources required to mechanize them. In that context the GLT approach is preferred. Consider the detection problem for n single degree-of-freedom sensors. In order to mechanize the detection decision for the GLT approach, an $(n-3)\times n$ matrix multiplication is required to obtain an $(n-3)$-dimensional vector of parity equation residuals. The components of p are then squared and summed and compared to a threshold. The mechanization for the TSE/MSR or maximum detectability approaches is similar, but the dimension $n-3$ for the GLT approach increases to n with no associated improvement in performance. Since detection must be done each time data are read, the benefits of the GLT approach, in terms of reduced computer throughput, are apparent.

The mechanization of the isolation decision for the GLT approach, assuming a symmetric sensor configuration, requires the computation of n $(n-3)$-dimensional vector dot products which are then squared and ranked in magnitude to identify the failed sensor. The mechanization for the TSE/MSR or maximum detectability approaches, on the other hand, requires only the ranking of the squared magnitude of the n parity equation residuals. Although the mechanization of the isolation decision is thus more complex for the GLT approach, the isolation decision is initiated only upon failure detection. The relatively low frequency of the detection event greatly reduces the significance of the difference in complexity of the isolation decision mechanization under various approaches. Thus, if mechanization requirements are the dominant consideration in selecting an FDI approach, the authors' view is that the GLT approach is preferred over the other approaches described.

Application

In this section, the GLT decision algorithms are applied to the problem of detecting and isolating a single failed sensor in each of three sensor configurations: a conical configuration of five SDOF sensors, a dodecahedron configuration of six SDOF sensors, and an octahedron configuration of four TDOF sensors. FDI performance criteria are defined, and Monte Carlo simulations are used to compare the FDI performance achieved for each of the sensor configurations.

The first configuration to be considered has the input axes of five SDOF sensors uniformly distributed about a cone with a central angle of 109.5 deg§ The sensor geometry matrix H_c is given by

$$H_c = \begin{bmatrix} 0.97204 & 0 & -0.23482 \\ -0.60075 & -0.77653 & -0.18997 \\ 0 & 0.47992 & 0.87731 \\ 0 & -0.47992 & 0.87731 \\ -0.60075 & 0.77653 & -0.18997 \end{bmatrix}$$

and the matrix V_c of parity equation coefficients is given by

$$V_c = \begin{bmatrix} 0.63245 & 0.51167 & 0.19544 & 0.19544 & 0.51167 \\ 0 & 0.37175 & 0.60150 & -0.60150 & -0.37175 \end{bmatrix}$$

Since there are five measurements $(n=5)$, there are two linearly independent parity equations. It can be verified that V_c satisfies Eqs. (7) and (19), so that Eq. (23) holds for $V=V_c$ and $H=H_c$. The decision function for detection is therefore given by Eq. (20). Furthermore, letting v_j denote the jth column of V_c, it can be verified that

$$v_j^T v_j = \frac{n-3}{n} = 0.4 \qquad \text{for all } j \tag{59}$$

Equation (59) implies uniform detectability. The decision function for isolation, therefore, is given by Eq. (22) where $V=V_c$, $p=V_c m$, and $j=1,2,\ldots 5$.

§The conical configuration described herein was proposed by Hamilton Standard, a Division of United Technologies.

The second configuration to be considered has the input axes of six SDOF sensors normal to be six nonparallel faces of a regular dodecahedron.[8] The sensor geometry matrix H_D is given by

$$H_D = \begin{bmatrix} 0.52573 & 0 & 0.85065 \\ -0.52573 & 0 & 0.85065 \\ 0.85065 & 0.52573 & 0 \\ 0.85065 & -0.52573 & 0 \\ 0 & 0.85065 & 0.52573 \\ 0 & 0.85065 & -0.52573 \end{bmatrix}$$

and the matrix V_D of parity equation coefficients is given by

$$V_D = \begin{bmatrix} 0.70711 & -0.31623 & -0.31623 & -0.31623 & -0.31623 & 0.31623 \\ 0 & 0.63246 & 0.19544 & 0.19544 & -0.51167 & 0.51167 \\ 0 & 0 & 0.60150 & -0.60150 & -0.37175 & -0.37175 \end{bmatrix}$$

Since there are six measurements ($n=6$), there are three linearly independent parity equations. It can be verified that V_D satisfies the constraints on the choice of a V matrix. The detection decision function is therefore given by Eq. (20). Furthermore, letting v_j represent the jth column of V_D

$$v_j^T v_j = \frac{n-3}{n} = 0.5 \qquad \text{for all } j \tag{60}$$

Thus, this configuration also exhibits the property of uniform detectability. Once again, the isolation decision function is given by Eq. (22) where $V=V_D$, $p=V_D m$, and $j=1,2,\ldots 6$.

The third configuration has the spin axes of each of four TDOF sensors normal to one of the faces of a half of a regular octahedron.¶ The sensors' measurement axes are oriented about the spin axes so that each measurement axis makes a 45 deg angle with one of the edges of the base of the semioctahedron. The sensor geometry matrix H_o is given by

$$H_o = \begin{bmatrix} 0.70711 & -0.40825 & 0.57735 \\ 0.70711 & 0.40825 & -0.57735 \\ 0.40825 & 0.70711 & 0.57735 \\ -0.40825 & 0.70711 & -0.57735 \\ -0.70711 & 0.40825 & 0.57735 \\ -0.70711 & -0.40825 & -0.57735 \\ -0.40825 & -0.70711 & 0.57735 \\ 0.40825 & -0.70711 & -0.57735 \end{bmatrix}$$

and the matrix V_o of parity equation coefficients by

$$V_o = \begin{bmatrix} 0.79057 & 0 & -0.15811 & 0.43198 & 0.15811 & 0.31623 & -0.15811 & -0.11575 \\ 0 & 0.79057 & -0.11575 & -0.15811 & 0.31623 & 0.15811 & 0.43198 & -0.15811 \\ 0 & 0 & 0.76590 & 0.06528 & -0.08278 & 0.53507 & 0.19585 & 0.27862 \\ 0 & 0 & 0 & 0.63964 & -0.16322 & -0.42451 & 0.58442 & 0.20607 \\ 0 & 0 & 0 & 0 & 0.68301 & -0.18301 & -0.18301 & 0.69301 \end{bmatrix}$$

Since there are eight measurements ($n=8$), there are five linearly independent parity equations. As in the previous cases, V_o was chosen to satisfy the constraints on the V matrix. The detection decision function is therefore given by Eq. (20). Once again,

$$v_j^T v_j = \frac{n-3}{n} = 0.625 \qquad \text{for all } j \tag{61}$$

so the octahedron configuration also provides uniform detectability. Furthermore, the matrix product $V_{ok}^T V_{ok}$ is the 5×2 partition of V_o associated with sensor k ($k=1,2,3,4$), does not depend on the index k. The isolation decision function is therefore given by Eq. (40) where $V=V_o$, $p=V_o m$, and $j=1,2,3,4$.

In order to compare the FDI performance which can be achieved by using the GLT decision function to detect and isolate a sensor failure in each of the sensor configurations, the detection decision function will be compared to a threshold which minimizes the probability of detection error. The detection error is defined as the sum of the probabilities of miss and false alarm. The use of an isolation threshold to reject false alarms and minimize wrong isolations will be omitted. Instead, the isolation decision function will remove a sensor whenever the detection threshold is exceeded. This choice of a detection threshold and the omission of an isolation threshold simplify the intended comparison. In practical applications, both detection and isolation thresholds would be required, and the determination of these thresholds would be based on a minimization of a cost function consisting of a weighted sum of the four unsuccessful outcomes of the decision structure as indicated in Fig. 1.

To assess detection performance for the decision process defined in the preceding paragraph, the performance function

$$C_D = [P_D + (1-P_{FA})]/2 \tag{62}$$

will be used. C_D implies a subjective judgment concerning the relative importance of detection in the presence of a failure and correct operation in the absence of a failure. To assess isolation performance for the decision process defined in the preceding paragraph, the performance function

$$C_I = [P_{CI} + (1-P_{FA})]/2 \tag{63}$$

will be used. Here again, a subjective judgment concerning the relative importance of detection and correct isolation in the presence of a failure and correct operation in the absence of a failure is implied.

¶The octahedron configuration described herein was proposed by Teledyne Systems Company.

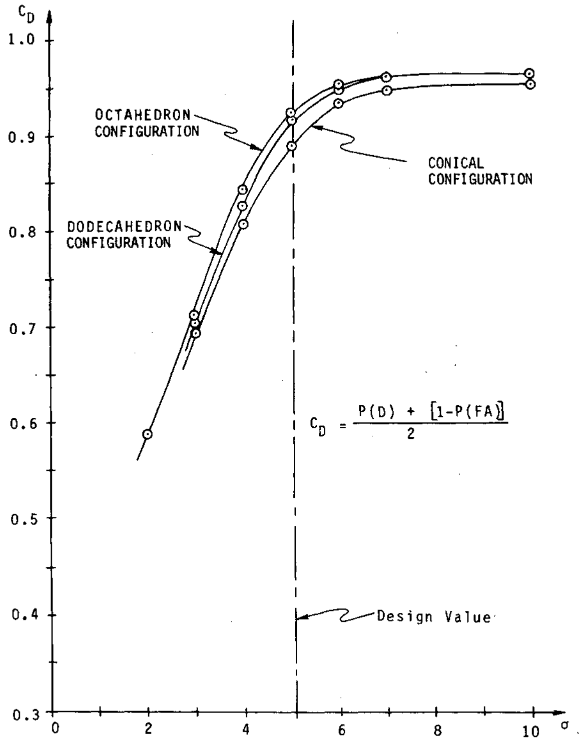

Fig. 2 Detection performance vs failure magnitude.

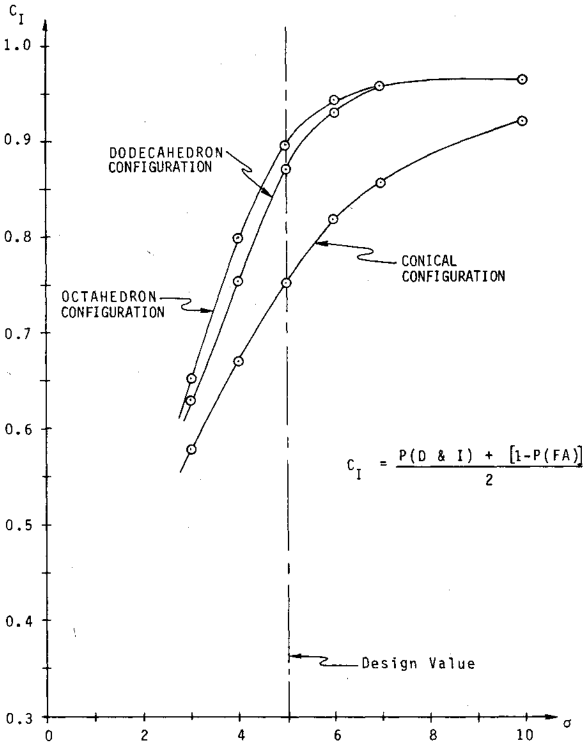

Fig. 3 FDI performance vs failure magnitude.

Monte Carlo simulations were used to compare the FDI performance, which can be achieved for the three sensor configurations using the GLT approach. In Figs. 2 and 3, the performance functions given in Eqs. (62) and (63) are plotted as functions of the bias failure magnitude. The measurement noise vector is assumed to have uncorrelated components with variance σ^2. The bias failure magnitude is represented in multiples of σ. The detection threshold for the data presented in Figs. 2 and 3 was chosen to minimize the probability of error for the detection decision in the presence of a 5σ bias failure.

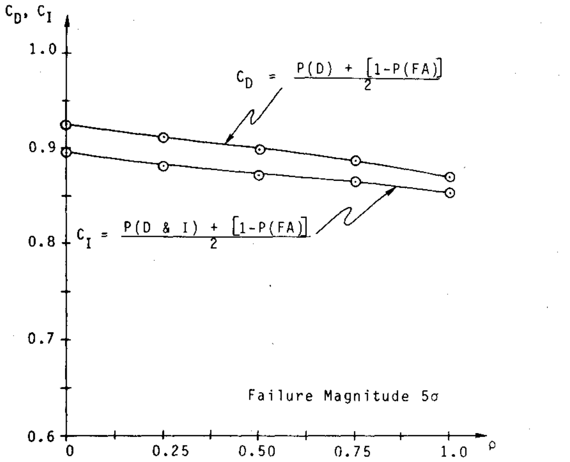

Fig. 4 Effect of correlated noise on algorithm designed for zero correlation.

Referring to Fig. 2, the number of sensor measurements and the configuration of the measurement axes are seen to have only a slight effect on detection performance. Although increasing the number of measurements increases the detection performance, the difference between the octahedron configuration of four TDOF sensors and the dodecahedron configuration of six SDOF sensors is not significant. The difference in the detection performance for the conical configuration of five SDOF sensors and the octahedron configuration of four TDOF sensors is less than 4% at the design value.

Referring to Fig. 3, it is apparent that a significant penalty is incurred in isolation performance when the conical configuration of five sensors is selected over the other sensor configurations considered. At the design value, this penalty amounts to as much as 15%. On the other hand, the isolation performance for the dodecahedron configuration of the six SDOF sensors is not significantly different from that which can be achieved for an octahedron configuration of four TDOF sensors.

The GLT decision functions for the octahedron configuration of four TDOF sensors were derived by assuming that the measurement noise observed in the two measurements derived from a TDOF sensor is uncorrelated. Monte Carlo simulations were used to determine the extent to which FDI performance using these decision functions is degraded if the noise is actually correlated. The results of these simulations are presented in Fig. 4. Figure 4 presents a plot of the two performance functions defined by Eqs. (62) and (63) vs the noise correlation coefficient ρ for a 5σ failure magnitude. The performance degradation across the spectrum of possible values of ρ is at most 5%. Since the actual value of ρ for a particular sensor is something less than one, the performance penalty incurred by assuming that $\rho=0$ in the derivation of the GLT decision functions should not exceed a few percent. The simplicity of the decision functions, which follows from the assumption that $p=0$, would seem to justify the small penalty incurred if this assumption is incorrect.

Conclusions

The GLT approach to FDI system design has been presented and the performance of such a system has been described in terms of the characteristics of the redundant sensor configuration. The GLT design can be applied to a wide range of sensor configurations and has been shown to provide performance equivalent to several more ad hoc approaches.

While the computational aspects of the GLT approach were not investigated in detail, the form of the detection decision function suggests that the GLT formulation may possess computational benefits relative to other approaches.

Although computational requirements may be a function of symmetries present in specific formulations, the choice of V presented makes p a sufficient statistic for the decision process and thus guarantees efficient computation in the general case. Features of the sensor configuration which lead to computational efficiency have been described.

The GLT approach provides a single, systematic design strategy for a wide class of sensor configurations. It has been found to be particularly useful in assessing the impact of sensor configurations upon FDI implementation and performance. It is anticipated that this design strategy will also prove valuable in the design of systems where the sensor configuration can undergo a significant change, either because of design evolution or because of a desire to provide an FDI system which allows hardware substitution.

Acknowledgment

The authors wish to acknowledge the contributions of M. B. Adams, whose experience in the application of hypothesis testing to failure detection and isolation problems proved useful throughout these studies, and J. E. Potter, whose paper[1] rewarded careful reading and whose proof was cited in the paper. This work was supported in part by contract number I-966872-9592 from the Boeing Aerospace Company.

References

[1] Potter, J.E. and Suman, M.C., "Thresholdless Redundancy Management with Arrays of Skewed Instruments, AGARD-OGRAPH-224, *Integrity in Electronic Flight Control Systems,* 1977, pp. 15-1 to 15-25.

[2] Wilcox, J.C., "Competitive Evaluation of Failure Detection Algorithms for Strapdown Redundant Inertial Instruments," *Journal of Spacecraft and Rockets,* Vol. 11, July 1974, pp. 525-529.

[3] Willsky, A.S. and Jones, H.L., "A Generalized Likelihood Ratio Approach to State Estimation in Linear Systems Subject to Abrupt Changes," IEEE Conference on Decision and Control, Nov. 1974.

[4] Carlson, N.A., "Statistical Basis for IMU Failure Detection and Isolation," Intermetrics Shuttle GN&C Memo 03-07, Jan. 1974.

[5] Wilcox, J., "Maximum Likelihood Failure Detection for Redundant Inertial Instruments," AIAA Paper 72-864, Stanford, Calif., Aug. 1972.

[6] VanTrees, H.L., *Detection, Estimation and Modulation Theory, Part I,* Wiley, New York, 1968.

[7] Harrison, J. and Gai, E., "Evaluating Sensor Orientations for Navigation Performance and Failure Detection," *IEEE Transactions on Aerospace and Electronic Systems,* Nov. 1977.

[8] Gilmore, J. and McKern, R., "A Redundant Strapdown Inertial Reference Unit (SIRU)," *Journal of Spacecraft and Rockets,* Vol. 9, Jan. 1972, pp. 39-47.

[9] Gai, E. and Harrison, J., "Redundant IMU Instrument Orientation Requirements for IUS," CSDL Rept. R-1076, March 1977.

[10] Chien, T.T. and Adams, M.B., "A Sequential Failure Detection Technique and Its Application," *IEEE Transactions on Automatic Control,* Oct. 1976.

[11] Harrison, J. and Chien, T.T., "Failure Isolation for a Minimally Redundant Inertial Sensor System," *IEEE Transactions on Aerospace and Electronic Systems,* May 1975.

[12] Musoff, H., "SIRU Utilization Volume I, Theory, Development and Test Evaluation," CSDL Rept. R-747, March 1974.

[13] Gai, E., Adams, M.B., and Walker, B.K., "Determination of Failure Thresholds in Hybrid Navigation," *IEEE Transactions on Aerospace and Electronic Systems,* Nov. 1976.

[14] Deckert, J.C., Desai, M.U., Deyst, J.J., and Willsky, A.S., "F-8 DFBW Sensor Failure Identification Using Analytic Redundancy," *IEEE Transactions on Automatic Control,* Oct. 1977.

[15] Schweppe, F., *Uncertain Dynamic Systems,* Prentice Hall, New York, 1973.

[16] Personal communication between the authors and James Potter.